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The Possibilities for the Application of the Logistic Model of Accumulation

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As a rule, under naturally operating conditions, especially in a closed environment, the product (understood in a general sense) cannot grow at equal rate. It is particularly obvious in a closed system maintaining limited resources necessary for the support of the concrete product's growth. In such a system, an initial rate of the product's increase gradually diminishes until its considerable decline and total cessation. Therefore, to model such a system, the differential equation of the growth of the population should be supplied with the factor expressing a straightly decreasing function. Hence the logistic model of accumulation is achieved. Meanwhile, the exponential function remains a special case in the model.

Sometimes it is reasonable to equate both the logistic and exponential functions. Then the coefficients of equivalence between the logistic and the exponential growth should be worked out.

The article presents a simplified way of the calculation of the regression coefficients of logistic equation with the use of the lowest square method. The paper offers the logistic regression equation of the gross domestic product of the Republic of Lithuania, which is equated with an analogical exponential equation.

Keywords: *population; product; logistic model of accumulation; exhaustible resources; compound interest; future value; lowest square method; regression equation; gross domestic product (GDP).*

Introduction

The model is understood as a reality representation reduced to essentials. Models are often employed for the investigation of the phenomena that occur around us: in nature, professional activity, or in the sphere of social relations. The mathematical or determined models are regarded as the most widespread ones (Мышкис, 2004).

Quantitative calculations of the growth of production, investment control, or money currents are usually based on the exponential models. However, by their nature, they cannot be accurate, especially, when forecasts must be extended into distant future. Basically, the law of decreasing limit resultativity operating in economic structures confirms this assumption (Bodie, Merton, 2000). Thus, the solution of such tasks requires though more complex yet more accurate logistic models. In fact, the problem of the practical application of the logistic models has been scantily investigated.

The aim of research is to investigate the models of accumulation by focusing on the specificity of the logistic models as well as to ascertain their merits and demonstrate their practical applicability by working out the logistic regression equation of a particular object.

Generalized determined models of the population product or, in certain cases of the capital accumulation serve as **the object of research**.

The analytical method of investigation has been employed which embraces the calculations of mathematical analysis and econometrics based on the possibilities offered by information technologies.

Natural exponential alteration: compound percentage

One of the kernel assumptions in the exploration of the product's exponential growth is the proportionality of the growth's rate to its quantity at each moment of time (Edwards, Penney, 1985). It will always remain so when a new product is involved into further reproduction, i.e. when it starts providing a new product on its own. In such a case, the product is said to increase naturally since by growing it offers an increasing increment on the basis of the same principle. Such an assumption is not difficult to understand when viewed in connection with the growth of the biological systems: the larger the developing system, the bigger its increment:

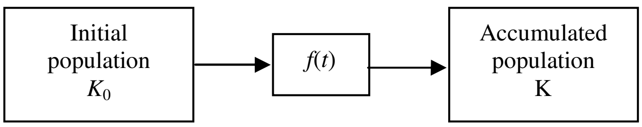

Figure 1. The structural scheme of the model of accumulation

Another example concerns capital: under certain conditions, capital gives interests that, in their turn, together with the old capital provide new interests, etc. (Obi, 1998; Valakevičius, 2001).

The determined models of accumulation that demonstrate a non-linear function of accumulation will be further analysed in the paper. Fig.1 offers the structural scheme of the model of population accumulation that is constructed on the basis of the following equation: $K = K_0 f(t)$.

Let's presume that K is the quantity of a certain product at the time moment t, and the rate of the product's growth is the rate of the alteration of the function that relates the altering t and K. In other words, the rate of the product's growth at every time moment is in proportion to its quantity. Let's also consider that the proportionality coefficient is constant and let's mark it by the letter i. Then

$$\frac{dK}{dt} = i \cdot K . \qquad (1)$$

The result is an elementary differential equation with the differentiated variable quantities. The equation solved and the initial conditions estimated, i.e. when $t = t_0$, the quantity of the product K is equal to its initial K_0, and when $K\big|_{t=0} = K_0$, the expression of the quantity of the product K is:

$$K = K_0 \cdot e^{i \cdot t} . \qquad (2)$$

If i is taken as the rate of interest and t is measured by the same time units as the time estimated in the interest rate, the equation (1) will show the natural exponential growth of the capital. To say more, this law is universal and may be applied for the modelling of other populations in order to both estimate their growth and vanishing (Rutkauskas, 2000).

Analogically, in the initial differential equation, let's take the proportionality coefficient equal to the natural logarithm of any number r $(r > 0, r \neq 1)$. Then, with $r = 1 + i$, the formula of compound percentage (i.e. of compound interest) is worked out:

$$K = K_0 \cdot (1+i)^t . \qquad (3)$$

It is an exponential function. It should be also noted that, when t is a natural number, the equation (3) serves as a formula to calculate a certain member of the geometrical progression. During the modelling of the product's growth by the given formula, it comes out that the product's growth is infinite. In reality, the forecasting of the product's increase in the distant future on the basis of this equation is inaccurate. At best, it may be used only in exceptional cases when the coefficient in the formula has been additionally corrected. No doubt, inaccurate financial forecasts may be explained by the imperfection of the formula (Misiūnas, 1997).

Logistic (i.e. limit) growth of the product

In the given case, the international term *logistics* is associated with the provision, i.e. with the possibility of the utilization of certain resources. As a rule, under naturally operating conditions, especially in a closed environment, the product cannot grow at equal rate. As has been mentioned it is particularly obvious in a closed system that possesses the necessary limited resources to maintain the growth of a particular product. As the initial rate of the product's increase gradually diminishes until considerable decline and total cessation, sometimes such a system is said to be the system of *the limited growth*.

The phenomenon of the limited growth is frequently observed in nature. It is especially distinct in the animal populations expanded in a closed territory. As long as the population is relatively small and possesses substantial food resources and large space, the rate of its increase is great.

Contrariwise, when the population increases, it possesses less and less space and food resources. Then the rate of its growth considerably diminishes (Tannenbaum, Arnold, 1995). A similar law of the so-called *decreasing limit resultativity* operating in the economic structures demonstrates that, in certain conditions, when the gross expenditure increases, the limit product (i.e. the rate of the product's growth) decreases. In economic terminology, such peculiarity is called the *limit capital efficiency*. Here the limit efficiency is the rate of return, which is expected from additional investments. With the increase of the investment amount, the limit investment efficiency decreases. It is so because the initial investments have been realized at the most favourable conditions, which caused large rates of income. Meanwhile, later investments are not so much effective and offer evenly smaller income (Pass, 1997, and others).

The operation of the mentioned law is especially obvious in agriculture. If initially, for the cultivation of the plot of constant size (for instance, one thousand hectares) only several workers are employed, and later their number considerably increases, after some time the limit work product (i.e. the rate of the work product's growth) will start decreasing. Economic literature maintains that the situation to be otherwise, the entire world might be nourished from the mentioned single plot (Wonnacott P., Wonnacott R., 1998). Undoubtedly, if the exponential law of growth operated all the time, the product's growth were unlimited and, during a long period, its manufacturing would grow to a desirable extent of amount.

Nevertheless it should be noted that, in most cases, the term *limit* employed in the theory of economics possesses another value, that is the value of rate, since it is associated with the mathematical derivative of the function estimating a certain phenomenon. The derivative is found out by the calculation of the limit (Van Horne, 1995).

Logistic (i.e. limit) future value of the product

As far ago as the 19^{th} century, when investigating the alteration of the biological systems, P.F. Verhülst suggested to supplement the differential equation of the growth of population with the multiplier possessing the shape of a linear decreasing function:

$$1 - \frac{K}{K_m} ; \qquad (4)$$

where K_m is the maximum (limit) value of the biological population or of any other product expressed by the units estimating its amount.

Let's apply the growth limiting multiplier (4) to the differential equation of the product's alteration (1). As before, let's take the proportionality coefficient equal to the natural logarithm of any number $r(r > 0, r \neq 1)$. The estimation of the initial conditions and the solution of the differential equation with respect to the product K results in the logistic (limit) future value of the product:

$$K = \frac{K_m \cdot K_0 \cdot r^t}{K_m + K_0(r^t - 1)}.$$

Here, presuming that $r = 1 + i$, the following formula is obtained:

$$K = \frac{K_m \cdot K_0 \cdot (1+i)^t}{K_m + K_0((1+i)^t - 1)}. \qquad (5)$$

It is the logistic (limit) future value of the product expressed by the rate of the growth percentage (Girdzijauskas, 2002).

Having divided both the nominator and the denominator of the equation's (9) right part by K_m and marked the ratio $\frac{K_0}{K_m}$ by the letter S_0 $\left(\frac{K_0}{K_m} = S_0,\ 0 \leq S_0 \leq 1\right)$, let's consider this ratio the coefficient of the initial congestion. In order to ascertain that time will be measured by the same units as the time estimated in the rate of the growth percentage, let's mark it by the letter n that, most frequently, expresses the entire periods of the percentage rate re-calculation. Thus re-worked, the future value of the limit alteration is:

$$K = \frac{K_0 \cdot (1+i)^n}{1 + S_0 \cdot ((1+i)^n - 1)}. \qquad (6)$$

The function of the product's logistic accumulation has been obtained, which demonstrates the relative expression, in other words, the coefficient of initial congestion.

It is important to maintain that, in case the product's maximum value K_m increases and approaches infinity $(K_m \to \infty)$, the coefficient of the initial congestion declines and its value approaches zero $(S_0 \to 0)$. Then, as one could expect, the formula (6) is no more than a conventional formula of compound percentage (3). The same will be obtained by having calculated the limit for the equation (5) when $K_m \to \infty$. On the other hand, if in the equation (6) $K_0 = K_m$, i.e. the initial product is equal to its maximum value, then the coefficient of the initial congestion S_0 will be equal to *1*. When in the equation (6) $S_0 = 1$, the result is the conclusion confirming the initial condition: $K_0 = K_m$.

Thus, the formula of compound percentage makes a separate case of the logistic function of accumulation when the maximum value of the product K_m is immensely large.

It should be also stressed that the obtained logistic function differs from the classical one whose expression is as follows:

$$K(x) = \frac{K \cdot e^{\lambda \cdot x}}{e^{\lambda \cdot x} + 1} \text{ or } K(x) = \frac{K}{1 + e^{-\lambda \cdot x}}. \qquad (7)$$

The latter function is defined in the entire set of real numbers and undergoes alteration within the interval (*0*; *K*). In other words, all the values of the function have been distributed between two horizontal lines $K(x) = 0$ and $K(x) = K$. Externally, the central part of the graph of the logistic function resembles a deformed Latin „*S*". The authors claim that, with the increase of the argument (i.e. time), the function (i.e. product) approaches the constant value equal to the quantity of the coefficient *K*. This peculiarity is the most essential for further generalizations. Fig.2 presents the graphs of this function when the coefficient *K* is constant and the values λ are different. With the increase of the coefficient λ when the argument is in the environment of zero, the rate of the function's growth further enlarges:

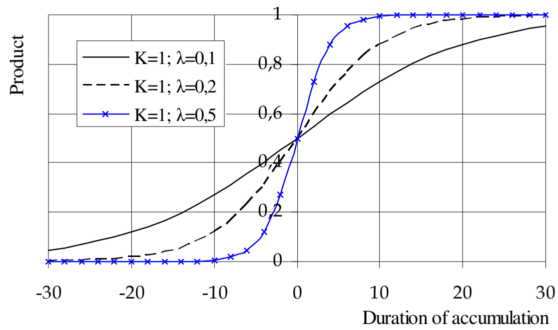

Figure 2. The graph of the logistic function

The alteration of the logistic function when the argument x is close to zero resembles the diagram of the exponential function (2). The authors maintain that, in this case, there is no direct relation between the exponential and logistic functions since there is no direct passage from one type of dependency to another.

Moreover, all the functions of the limited growth whose character of alteration is similar to the alteration of the logistic function (7) are also called logistic. The logistic functions of the type of the formula (6) that are more fit for the modelling of the financial phenomena will be discussed further as well as the possibilities of the exponential and logistic function application.

Let's compare the diagrams of the exponential (7) and logistic (6) accumulation (i.e. the graphs of the future values). Fig. 3 presents the diagrams that illustrate the dependency of the product's future value on time. Here the interest rate of the graphically illustrated functions makes 20% and the initial congestion S_0 alters from 0 to 20% (i.e. the exponential variant shows the absence of congestion). At the initial moment, the product's value is equal to one ($K_0 = 1$). It means that the initial product is equal to one and that other values are expressed by the initial product.

No doubt, the coefficient of the initial congestion S_0 significantly affects the alteration of the limit function. With the decrease of the initial congestion coefficient, when the initial K_0 product is constant, the limit of this function increases. If the coefficient S_0 gradually declines (i.e.corresponding the curve in the graph when $S_0 = 0$), the limit becomes infinite and the limit function, in its turn, grows into the formula of compound percentage. Obviously, with the increase of the initial congestion coefficient, the product's future value decreases despite other parameters of the logistic function remaining unchanged.

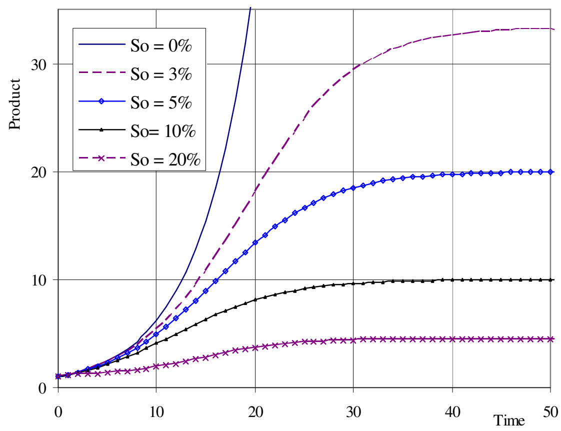

Figure 3. The product's dependency on time when the values of the initial congestion are various and when *i=0,2, Ko=1*

The diagram shows that, at the beginning (when *n* values are low), all the functions (logistic and conventional) do sufficiently overlap. To say more, a rather inconsiderable value difference (e.g. about 5%) remains for a longer period in case the percentage rate is smaller. Later on the rise of the limit function graphs slackens until its complete cessation.

The mentioned peculiarity of the diagram can be easily observed in the analytical investigation of the logistic function. The search for $\lim_{n\to\infty} K$ leads to the finding the indeterminacy ∞/∞. Here the application of the Liopital's rule results in the following:

$$\lim_{n\to\infty}\frac{K_0(1+i)^n}{1+S_0\left((1+i)^n-1\right)}=\lim_{n\to\infty}\frac{K_0(1+i)^n\ln(1+i)}{0+S_0\left((1+i)^n\ln(1+i)-0\right)}=\frac{K_0}{S_0}=K_m$$

It should be noted that the semblance of value at the initial moment graphically illustrates the fact discussed above, i.e. the exponential function (3) making a separate case of the logistic function (6).

Limit interest

It is time to discuss a particular product, i.e. capital, and its increase in a certain period of time, i.e. interest. The compound interest as well as the limit interest point to the difference between the accumulated capital K and the initial capital K_0:

$$P = K - K_0\,,$$

where P is the interest accumulated during a certain period.

The expression of the future capital value of the limit alteration (5) allows for the following:

$$P=\frac{K_m\cdot K_0\cdot(1+i)^t}{K_m+K_0\left((1+i)^t-1\right)}-K_0\,.$$

Then:

$$P=K_0\frac{(K_m-K_0)\left((1+i)^n-1\right)}{K_m-K_0\left((1+i)^n-1\right)}\,.$$

The numerator and the denominator having divided by K_m, the result is:

$$P=K_0\frac{(1-S_0)\left((1+i)^n-1\right)}{1+S_0\left((1+i)^n-1\right)}\,. \qquad (8)$$

Another division by $\left((1+i)^n-1\right)$ marked as follows: $\frac{1}{(1+i)^n-1}=u_n$ leads to such expression of the limit interest:

$$P=K_0\frac{1-S_0}{u_n+S_0}\,.$$

When the congestion S_0 is constant, the quantity of interest depends on the accumulation factor u_n, which, in its turn, is the function of the interest rate and accumulation duration. It decreases both with the increase of the interest rate and of accumulation time. The quantity of the interest also has its limit, which is equal to $K_0(1-S_0)/S_0$.

Application of limit accumulation models

Practically, the problem of the logistic (limit) accumulation of the capital has not yet been investigated. It was determined by several important reasons. First of all, no adequate models have been worked out. Obviously, the logistic models are much more complete than the exponential ones and therefore their analysis is possible only with the invocation of information technologies. Finally, the logistic models should be mostly applied for the closed economic systems; however, the mentioned condition is fully realized rather rarely.

The statistic analysis of the long-term economic development of some systems allows for the conclusion that, in certain periods, there exist the indications pointing to the fact of approaching a certain limit. Such conclusions are found in scanty publications on the mentioned problem (Ferreira, Eidelman, 1998).

Let's consider the case of the economic structure in Brazil. For a long time, it was isolated from external resources. Carlos Feu Alvim based his analysis of the GDP in Brazil in 1947-1992 on this very fact. By having employed the phenomenological methodology and based his research on the analysis of the national indices, the author arrived at the conclusion that, at the end of the investigated period, the accumulated capital hardly reaches 7% of the value, which was prognosed by the exponential model. In 1998 another more profound investigation was carried out by employing similar methodology and operating with new additional data (until 1997).

During their investigation Omar Compos Fereira and Carlos Feu Alvim strove to discover the structural reasons of the end of 'the economic miracle' in Brazil. The chief hypothesis concerned the capital being the factor that limited the growth of Brazil's economic level (here manpower was not sufficiently exploited). The given conclusion is that up to the seventh decade the economic structure of Brazil may be described by the logistic model, which reveals that the rate of the economic growth is stemmed by the limit capital. The researchers offered an insignificantly modified variant of the classical logistic function (7). It goes without saying that the employment of the logistic function (6) brings more precise forecasting results.

Determination of the regression coefficients

In the application of the logistic models of accumulation, the determination of the regression coefficients comes to be one of the most important and problematic tasks.

Let's define the conditions under which one model of capital accumulation (CAM) might be replaced by another, i.e. more elementary model. When the maximum value of the capital limits its growth insignificantly, the exponential analytic model may be employed. As has been observed above, the values calculated by both models at the beginning of the growth period are rather close. It has been also noted that the exponential model solely makes a separate case of the logistic model. Therefore having the time line of the values of the capital and knowing that the variable quantities operate according to the law of the limit growth allows for the description of the initial values of the time line by the exponential model. Furthermore, the coefficients of the latter model may be used in the construction of the logistic model.

By applying the above mentioned conclusions, the logistic model coefficients K_0 and i will be calculated with the help of the lowest square method and according to the exponential model (Boguslauskas, 2004). The expression of the exponential CAM $y = a(1+i)^x$ will serve as the regression equation:

$$\hat{y}_1 = a(1+i)^{x_1},$$

$$\hat{y}_2 = a(1+i)^{x_2},$$

....

$$\hat{y}_n = a(1+i)^{x_n}.$$

The lowest square method allows for the description of the differences between the theoretical and experimental values and for the structuring of the sum of deviation squares:

$$u = (\hat{y}_1 - y_1)^2 + (\hat{y}_2 - y_2)^2 + \ldots + (\hat{y}_n - y_n)^2$$

By having inserted the modelled y values of the deviations and replaced $(1+i)$ by r $(1+i = r)$ the function is described as follows:

$$u = (a \cdot r^{x_1} - y_1)^2 + (a \cdot r^{x_2} - y_2)^2 + \ldots + (a \cdot r^{x_n} - y_n)^2 \quad (11)$$

The function (11) possesses two variable quantities, i.e. a and r.

Let's apply the obtained function to the calculation of the regression coefficients in the solution of a particular problem. Table 1 presents the dynamics of the GDP in Lithuania in 1992-2003 (in mln Lt; Department of Statistics, 2004). The data of the period from 1992 to 1995 is taken for the determination of the coefficients of the exponential function:

Table

Lithuania's GDP in 1992-2003

Years	1992	1993	1994	1995	1996	1997
GDP	3406	11590	16904	25568	32290	39378
Years	1998	1999	2000	2001	2002	2003
GDP	44377	43359	45526	48379	51633	55737

Relying on the mentioned data of the initial four years, Fig.4 shows the dependency of the sum of deviation squares on the growth rate coefficient r at different a values. Fig. 4 demonstrates that this function has its minimum when r is between 1.5 and 2.0 and a is close to 6000.

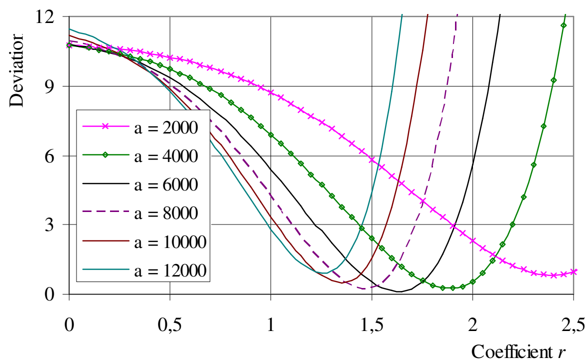

Figure 4. The dependency of the sum of deviation squares on the growth rate coefficient r at different a values

In order to find out the precise minimum of the function as well as the coefficients in-search it is necessary to calculate the function's derivatives with respect to separate variable quantities and to equate them to zero.

The calculation of the derivatives leads to the following:

$$\begin{cases} \dfrac{\partial u}{\partial a} = 2\cdot(a\cdot r^{x_1} - y_1)\cdot r^{x_1} + 2\cdot(a\cdot r^{x_2} - y_2)\cdot r^{x_2} + \ldots \\ \quad \ldots + 2\cdot(a\cdot r^{x_n} - y_n)\cdot r^{x_n}; \\ \dfrac{\partial u}{\partial r} = 2\cdot(a\cdot r^{x_1} - y_1)\cdot a\cdot x_1\cdot r^{x_1 - 1} + \\ \quad + 2\cdot(a\cdot r^{x_2} - y_2)\cdot a\cdot x_1\cdot r^{x_2 - 1} + \ldots \\ \quad \ldots + 2\cdot(a\cdot r^{x_n} - y_n)\cdot a\cdot x_n\cdot r^{x_n - 1}. \end{cases}$$

The lowest values should be found among the solutions of the system: $\begin{cases} \dfrac{\partial u}{\partial a} = 0; \\ \dfrac{\partial u}{\partial r} = 0, \end{cases}$. The rearrangements and the solution of each equation of the system with respect to the unknown quantity result in:

$$\begin{cases} a = \dfrac{\sum_{j=1}^{n}(r^{x_j} \cdot y_j)}{\sum_{j=1}^{n} r^{2x_j}}; \\ a = \dfrac{\sum_{j=1}^{n}(x_j \cdot y_j \cdot r^{x_j})}{\sum_{j=1}^{n}(x_j \cdot r^{2x_j})}. \end{cases}$$

The equalization of the equations' right parts and certain slight modifications lead to the following:

$$\sum_{j=1}^{n}(r^{x_j} \cdot y_j) \cdot \sum_{j=1}^{n}(x_j \cdot r^{2x_j}) - \sum_{j=1}^{n} r^{2x_j} \cdot \sum_{j=1}^{n}(x_j \cdot y_j \cdot r^{x_j}) = 0 \quad (12)$$

The analytical solution of this equation is rather complicated, especially when n is considerably higher than 2.0. The most convenient way of its solution is found in the use of information technologies. The authors see one of such solutions in the creation of the iteration cycle during which the value of the regression coefficient is altered until it satisfies the formed equation at necessary precision.

The solution of the obtained equation and the finding of the coefficient allows for the calculation of the second regression coefficient a. The value of the coefficient a is determined due to one of the system's equations. In the analysed example, the values are as follows: $r = 1.628714$ and $a = 6049.49$.

In the estimation of the obtained values of the coefficient, the specificity of the problem should be accentuated. In fact, the statistic data reflects the situation of the transitional economic period. On the other hand, the values of the regression coefficients r and a depend not only on the statistic data acquired during particular observations but also on the number of the observed elements employed in the concrete calculation.

When possessing the necessary r and a coefficients, it is important to determine the final logistic model's coefficient K_m. Let's define $K = Y$, and $K_m = z$. As before, let's presume that $1 + i = r$ and $K_0 = a$. Then the logistic function of accumulation will read as follows:

$$Y = \frac{a \cdot z \cdot r^x}{z + a \cdot (r^x - 1)}.$$

From this it is found out that the values of the function conforming to the arguments $x_1, x_2, ..., x_n$ are:

$$\hat{Y}_1 = \frac{a \cdot z \cdot r^{x_1}}{z + a \cdot (r^{x_1} - 1)}$$

$$\hat{Y}_2 = \frac{a \cdot z \cdot r^{x_2}}{z + a \cdot (r^{x_2} - 1)}$$

....

$$\hat{Y}_n = \frac{a \cdot z \cdot r^{x_n}}{z + a \cdot (r^{x_n} - 1)}$$

The unknown regression coefficient z may be calculated by employing the lowest square method:

$$v = (\hat{Y}_1 - Y_1)^2 + (\hat{Y}_2 - Y_2)^2 + ... + (\hat{Y}_n - Y_n)^2$$

Finally, by inserting the modelled values of the variable quantities the required function is obtained:

$$v = (\frac{a \cdot z \cdot r^{x_1}}{z + a \cdot (r^{x_1} - 1)} - Y_1)^2 + $$
$$+ (\frac{a \cdot z \cdot r^{x_2}}{z + a \cdot (r^{x_2} - 1)} - Y_2)^2 + ... +$$
$$+ (\frac{a \cdot z \cdot r^{x_n}}{z + a \cdot (r^{x_n} - 1)} - Y_n)^2.$$

In order to discover the minimum of the function it is necessary to calculate the function's derivate according to z and to equalize it to zero:

$$\frac{dv}{dz} = 2 \cdot (\frac{a \cdot z \cdot r^{x_1}}{z + a \cdot (r^{x_1} - 1)} - Y_1) \cdot \frac{a^2 \cdot r^{x_1} \cdot (r^{x_1} - 1)}{(z + a \cdot (r^{x_1} - 1))^2} +$$
$$+ 2 \cdot (\frac{a \cdot z \cdot r^{x_2}}{z + a \cdot (r^{x_2} - 1)} - Y_2) \cdot \frac{a^2 \cdot r^{x_2} \cdot (r^{x_2} - 1)}{(z + a \cdot (r^{x_2} - 1))^2} + ... +$$
$$+ 2 \cdot (\frac{a \cdot z \cdot r^{x_n}}{z + a \cdot (r^{x_n} - 1)} - Y_n) \cdot \frac{a^2 \cdot r^{x_n} \cdot (r^{x_n} - 1)}{(z + a \cdot (r^{x_n} - 1))^2} = 0.$$

Insignificant modifications allow for a reduced expression of the equation:

$$\sum_{j=1}^{n}((\frac{a \cdot z \cdot r^{x_j}}{z + a \cdot (r^{x_j} - 1)} - Y_j) \cdot (\frac{a^2 \cdot r^{x_j} \cdot (r^{x_j} - 1)}{(z + a \cdot (r^{x_j} - 1))^2})) = 0$$

In order to find out the regression coefficient z the equation must be solved and the value z, by which the equation is satisfied, must be obtained. It is convenient to solve such an equation with the help of the digital method. Digital methods are iteration procedures where the results achieved during each step are compared with the earlier achieved ones to choose the best solution.

Thus, the choice of the accuracy of the variable coefficient z and of the required number of the cycle's iterations allows for the obtaining of the sought value. In this case, the determined z value may be considered optimal.

The data of the time line used for the determination of the coefficient z also embraces the data employed for the determination of the coefficients a and r. In other words, the latter data overlaps with the former. Then the obtained coefficients important for the statistics of the GDP in Lithuania are inserted into the logistic regression equation:

$$K = \frac{56877{,}5 \cdot 6049{,}49 \cdot 1{,}629^t}{56877{,}5 + 56877{,}5 \cdot \left(1{,}629^t - 1\right)} .$$

The statistic data of the GDP and the logistic regression curve are presented in Fig. 5:

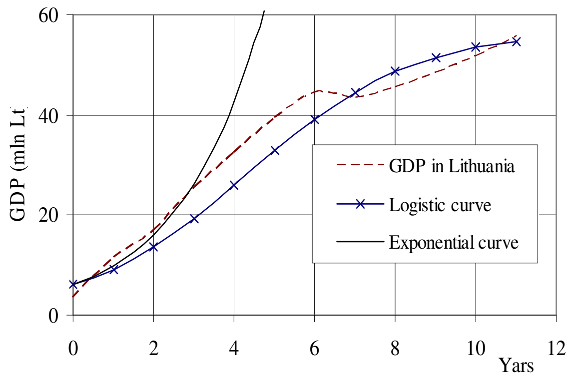

Figure 5. Regression curves of Lithuania's GDP

Fig. 5 also shows the operation of the exponential function obtained on the basis of the initial data:

$$K = 6049{,}489 \cdot 1{,}629^{\,t} .$$

It apparently demonstrates that the forecasting directed into the remote future causes considerable deviations (e.g. in the middle of the interval, the error exceeds 100%).

Conclusions

Logistic (limit) models are applied for the investigation of the alteration of some kinds of population. The majority of populations, especially capital, distinguish by their reproductivity, i.e. the capacity to get restored, to renew, and expand (multiply). In the case of the capital, it manifests itself in the interest that participates in the creation of the capital of new generation (together with the initial capital). Therefore, until restrictions do not occur, it might be considered that the capital is growing in a constant rate. The exponential models are used for the modelling of the alteration of the permanently growing product. However, such models are not always sufficiently precise and convenient for practical use. The analysis of the models of accumulation has lead to the following conclusions:

- The logistic models allow for the modelling of the development of the populations whose growth is limited by the insufficiency of resources. The logistic models of accumulation reflect the dynamics of the population's (capital's) growth more precisely;
- The exponential model makes a separate case of the logistic (limit) model;
- The expression of capital in a closed system demonstrates the slackening character and has its limit equal to its maximum value;
- The lowest square method used to determine the regression coefficients may be applied gradually. For this purpose, it is convenient to employ the exponential function as a separate case of the logistic function;
- The logistic models may be used in the construction of the regression coefficient of a particular object.

Such proposition is worked out on the logistic regression equation of the GDP in Lithuania that was made on the basis of the statistic data from twelve latter years.

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Logistinių kaupimo modelių taikymo galimybės

Santrauka

Dauguma besivystančių populiacijų, priklausomai nuo jų prigimties, naudoja kokius nors išteklius. Ištekliai savo ruožtu, priklausomai nuo jų santykio su pačia populiacija, gali būti tiek baigtiniai, tiek begaliniai. Logistiniai kaupimo modeliai yra paremti senkančių išteklių įtaka populiacijos augimo procesui. Išteklių ribotumas, savo prigimtimi būdamas vienas svarbiausių daugelio sistemų vystymosi veiksnių, dažnai per menkai vertinamas, ir Tai yra ne tik dėl to, kad dauguma išteklių yra sunkiai išmatuojami, bet ir dėl to, kad, trūkstant patikimų ir patogių prognozavimo priemonių, kartais neįmanoma net apytikriai numatyti rezultatus.

Darbe sutelktas dėmesys į specifines populiacijas, t.y. populiacijas, gebančias augti natūraliai, į tokias, kurios augdamos pačios duoda pagal tą patį principą didėjantį prieaugį. Natūralus augimas – tai toks augimo greitis, kai kiekvienu laiko momentu jis proporcingas populiacijos dydžiui: kuo didesnė populiacija, tuo ji greičiau auga. Tokioms populiacijoms gali būti priskiriamas kapitalas, pinigų srautai ar kitas panašias savybes turintis produktas. Tai leidžia užrašyti paprastas diferencialines lygtis, įgalinančias sudaryti patogius populiacijų vystymosi modelius.

Jei K – tam tikro produkto dydis laiko momentu t, o to produkto augimo greitis yra kintamuosius t ir K siejančios funkcijos kitimo greitis, tai, laikydami, kad kintamuosius sieja proporcingumo koeficientas i, gausime, kad $dK/dt = i \cdot K$. Išsprendę šią lygtį ir įvertinę pradines sąlygas, kad laiko momentu $t = t_0$ produkto dydis K lygus pradinei jo reikšmei K_0, randame produkto kiekio K išraišką

$$K = K_0 \cdot e^{i \cdot t}.$$

Tuo pačiu principu galima sudaryti ir išteklių ribotumą įvertinančius augimo (logistinius) modelius. Logistinių modelių yra įvairių. Kai kurie jų plačiai taikomi biologinių sistemų tyrimui. Tačiau ekonominių reiškinių nagrinėjimui šie modeliai sunkiai pritaikomi. Jų trūkumas – nėra natūralaus perėjimo į sudėtinių procentų išraišką.

Statistiškai nagrinėjant kai kurių sistemų ilgalaikį ekonominį vystymąsi, pastebėta, kad atskirais periodais egzistuoja artėjimo prie tam tikros ribos požymiai. Izoliuotoje aplinkoje platesnės apimties ekonominį eksperimentą atlikti yra sudėtinga, todėl teko nagrinėti pasaulinę patirtį, ieškoti realių praktinių pavyzdžių. Žinoma, kad Brazilijos ekonomika ilgą laiką buvo izoliuota nuo išorinių išteklių. Įvertinus tą faktą, Karlas Alvimas (Carlos Feu Alvim) ir Omaras Fereira (Omar Compos Ferreira) atlikto Brazilijos BVP augimo tyrimus. Naudodami fenomenologinę metodologiją ir remdamiesi nacionalinių rodiklių analize, autoriai priėjo išvadą, kad laikotarpio pabaigoje sukauptasis kapitalas siekia vos 7 proc. tos reikšmės, kuri buvo prognozuojama remiantis rodikliniu modeliu. Buvo ieškoma struktūrinių "Brazilijos ekonominio stebuklo" pabaigos priežasčių. Pagrindinė hipotezė ta, kad ekonomikos augimą ribojantis veiksnys yra kapitalas (darbo jėga Brazilijoje buvo išnaudojama ir taip nepakankamai). Taigi čia prieinama išvada, kad nagrinėjamo laikotarpio ekonomika gali būti aprašoma logistiniu modeliu, kur augimo tempą stabdantis veiksnys yra ribinis kapitalas. Kapitalo augimo modeliavimui tyrimo autoriai siūlė mažai modifikuotą klasikinės logistinės funkcijos variantą.

Šiame darbe naudojamas patobulintas logistinis augimo modelis. Jis turi pertvarkytus koeficientus ir yra bendriausias tokio tipo populiacijų augimo modelių atvejis:

$$K = \frac{K_m \cdot K_0 \cdot (1+i)^t}{K_m + K_0\left((1+i)^t - 1\right)}$$

(čia K_0 – pradinė populiacija, išreikšta jos kiekį įvertinančiais vienetais, K_m – maksimali (ribinė) populiacijos reikšmė, i – augimo norma, t – augimo trukmė, išreikšta tais pat laiko vienetais kaip ir augimo laikas). Plačiai iki šiol kapitalo augimui modeliuoti taikyta sudėtinių procentų taisyklė $K = K_0 \cdot (1+i)^t$ yra atskirasis šio logistinio modelio atvejis. Parodoma, kad tipinė matematinėje analizėje vartojama logistinė funkcija $K = K_m / \left(1 + e^{-\lambda x}\right)$ skiriasi nuo nagrinėjamos funkcijos ir nėra šios funkcijos kuris nors atskirasis atvejis. Reikia pažymėti, kad mūsų siūlomas patobulintas logistinis augimo modelis duoda tikslesnius prognozės rezultatus, nei klasikinis.

Vienas svarbiausių sistemos augimo rodiklių yra per tam tikrą laiką sukauptos palūkanos. Logistinis modelis taip pat leidžia apskaičiuot tokias palūkanas. Ribinės palūkanos P yra sukauptojo K ir pradinio K_0 kapitalų skirtumas:

$$P = K - K_0.$$

Sutrumpinta tokių palūkanų išraiška yra

$$P = K_0 \frac{1 - S_0}{u_n + S_0},$$

kur $S_0 = K_0 / K_m$ – pradinio prisotinimo koeficientas, o $u_n = 1/\left((1+i)^n - 1\right)$. Esant pastoviam prisotinimui S_0, palūkanų dydis priklauso nuo kaupimo veiksnio u_n, kuris savo ruožtu yra palūkanų normos ir kaupimo trukmės funkcija. Jis mažėja, tiek didėjant palūkanų normai, tiek kaupimo laikui. Palūkanų dydis taip pat turi ribą, lygią

$$K_0 (1 - S_0)/S_0.$$

Straipsnyje pateikiama ne tik teorinė augimo modelių, apibrėžtų išteklių išsenkamumu, analizė, bet ir tokių modelių praktinio taikymo galimybių įvertinimas. Taikant logistinius kaupimo modelius, viena svarbiausių ir problemiškiausių užduočių yra regresijos koeficientų nustatymas. Dažniausiai tam naudojamas mažiausių kvadratų meto-

das. Uždavinio sprendimo sunkumas tas, kad šis modelis yra netiesinis ir turi ne mažiau kaip tris neapibrėžtus koeficientus: K_0, K_m ir i. Todėl mažiausių kvadratų metodą logistinio modelio regresijos koeficientams nustatyti siūloma taikyti palaipsniui. Pradiniu laiko momentu populiacijos reikšmės, apskaičiuotos remiantis logistiniu ir eksponentiniu modeliu, yra labai artimos. Todėl patogu, panaudojus atskirąjį logistinės funkcijos atvejį – eksponentinę funkciją, nustatyti pradinius logistinės funkcijos koeficientus: pradinį populiacijos dydį K_0 ir augimo greičio koeficientą i. Toliau, įrašius šiuos duomenis į bazinę išraišką, nesudėtinga apskaičiuoti ir ribinę (didžiausią) populiacijos reikšmę.

Teorinių apibendrinimų teisingumas pagrindžiamas konkretaus objekto regresijos lygties sudarymu. Tam panaudojami 1992–2003 metų Lietuvos BVP statistiniai duomenys. Pirmiausia eksponentinės funkcijos koeficientams nustatyti imami pirmųjų ketverių metų duomenys. Jais remiantis, sudaryta nuokrypų kvadratų sumos priklausomybė nuo augimo greičio koeficiento, esant skirtingoms pradinėms populiacijos reikšmėms. Įsigilinę pastebime, kad ši funkcija turi minimumą, esantį tarp 1,5 ir 2, kai pradinė populiacija yra artima 6 000. Ieškant tikslaus šios funkcijos minimumo, apskaičiuojamos funkcijos išvestinės atskirų kintamųjų atžvilgiu ir jos prilyginamos nuliui. Palyginę dešiniąsias lygčių puses ir atlikę nežymius pertvarkymus, gauname šią lygtį:

$$\sum_{j=1}^{n}(r^{x_j}\cdot y_j)\cdot\sum_{j=1}^{n}(x_j\cdot r^{2x_j})-\sum_{j=1}^{n}r^{2x_j}\cdot\sum_{j=1}^{n}(x_j\cdot y_j\cdot r^{x_j})=0\cdot$$

Analitinis šios lygties sprendimas yra gana keblus, ypač kai laipsnio rodiklis n daug didesnis nei 2. Patogiausia tokius uždavinius spręsti naudojantis informacinėmis technologijomis. Vienas iš galimų sprendimo būdų – iteracinio ciklo sudarymas, kurio metu kiekvienoje iteracijoje keičiama vieno iš regresijos koeficientų reikšmė tol, kol ji galiausiai reikiamu tikslumu atitinka sudarytąją lygtį. Antrojo koeficiento reikšmė nustatoma išsprendus vieną iš pradinės sistemos lygčių, kurioje pirmojo koeficiento reikšmė jau yra žinoma. Tokiu būdu mūsų nagrinėjamame pavyzdyje šios reikšmės yra: augimo greičio koeficientas – 1,628714, o pradinio kapitalo reikšmė – 6049,49 mln. litų.

Turint minėtus du koeficientus, apskaičiuojamas ir paskutinis logistinio modelio koeficientas K_{m}. Jis randamas, logistinės lygties išvestinę prilyginus nuliui

$$\sum_{j=1}^{n}((\frac{a\cdot z\cdot r^{x_j}}{z+a\cdot(r^{x_j}-1)}-Y_j)\cdot(\frac{a^2\cdot r^{x_j}\cdot(r^{x_j}-1)}{(z+a\cdot(r^{x_j}-1))^2}))=0$$

Ir ankstesniųjų, ir šios lygties sprendimui taikome apytikslius metodus. Atlikę reikalingus apskaičiavimus, gauname, kad paskutiniojo koeficiento reikšmė yra 6049,49.

Tada logistinis Lietuvos BVP augimo modelis būtų:

$$K=\frac{56877{,}5\cdot 6049{,}49\cdot 1{,}629^{\,t}}{6049{,}49+56877{,}5\cdot\left(1{,}629^{\,t}-1\right)}.$$

Vertinant gautąsias koeficientų reikšmes reikia pabrėžti uždavinio specifiką: statistiniai duomenys atspindi pereinamojo ekonominio laikotarpio situaciją. Kita vertus, regresijos koeficientų K ir i reikšmės priklauso ne tik nuo statistinių duomenų, gautų atskirų stebėjimų metu, bet ir nuo stebinių skaičiaus, panaudotų konkrečiame skaičiavime.

Apibendrinant daroma išvada, kad logistiniai kaupimo modeliai tiksliau atspindi populiacijos (kapitalo) augimo dinamiką ir gali būti taikomi ten, kur reikia įvertinti senkančius išteklius. Eksponentinis modelis yra tik atskiras logistinio modelio atvejis. Mažiausių kvadratų metodą logistinio modelio regresijos koeficientams nustatyti galima taikyti palaipsniui.

Raktažodžiai: *populiacija, produktas, modelis, senkantys ištekliai, sudėtinės palūkanos, logistinis augimas, būsimoji vertė, mažiausių kvadratų metodas, regresijos lygtis, bendrasis vidinis produktas (BVP)*

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