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Modelling Financial Market Volatility Using Asymmetric-Skewed-ARFIMAX and -HARX Models

Wen Cheong Chin[1], Min Cherng Lee[2], Grace Lee Ching Yap[3]

[1]Multimedia University
63100 Cyberjaya, Selangor, Malaysia
E-mail. wcchin@mmu.edu.my

[2]Universiti Tunku Abdul Rahman
43300, Kajang, Selangor, Malaysia
E-mail. leemc@utar.edu.my

[3]University of Nottingham Malaysia Campus
43500 Semenyih, Selangor, Malaysia
E-mail. grace.yap@nottingham.edu.my

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This study aims to propose an improved modelling framework for high frequency volatitliy in financial stock market. Extended heterogeneous autoregressive (HAR) and fractionally integrated autoregressive moving average (ARFIMA) models are introduced to model the S&P500 index using various realized volatility measures that are robust to jumps. These measures are the tripower variation volatility, and the realized volatilities integrated with the nearest neighbor truncation (NNT) approach, namely the minimum and the median realized volatilities. In order to capture volatility clustering and the asymmetric property of various realized volatilities, the HAR and ARFIMA models are extended with asymmetric GARCH threshold specification. In addition, the asymmetric innovations of various realized volatilities are characterized by a skewed student-t distribution. The empirical findings show that the extended model returns the best performance in the in-sample and out-of-sample forecast evaluations. The forecasted results are used in the determination of value-at-risk for S&P500 market.

Keywords: *Realized Volatility, Heterogeneous Autoregressive Model, Value at Risk.*

Introduction

The availability of high frequency data in financial time series has great contribution to the accuracy of volatility estimations especially in the application of portfolio investment (Cervello-Royo *et al.*, 2015; Goumatianos *et al.*, 2013; Vella & Ng, 2014) and risk management (Dionne *et al.*, 2015; Liu & Tse, 2015; Louzis *et al.*, 2014). One of the immediate applications of estimated volatility is the determination of market risk using the value-at-risk. In recent years, many stock market investment institutes have added risk management units to identify risks and implement strategies to overcome the potential risks in the globalized financial stock markets. Thus, financial risk management (Hammoudeh & McAleer, 2015) has become a crucial component in the stock market investments where failure to manage the market risks may result in severe losses to their investments. Owing to that, a reliable and accurate risk management analysis is highly desired to measure the potential market risk in the current stock market investment strategies. In general, the types of data and statistical models have direct impact on the accuracy of market risk management. It is well-known that the stock market related industries are highly driven by data. The information of inter-daily data is no longer able to accommodate the massive amount of large-scale trading activities.

One of the important early studies of high frequency data was conducted by Andersen and Bollerslev (1998) which is commonly known as the realized volatility (RV). The theoretical properties of RV can be found in Andersen *et al.* (2003) and Barndorff-Nielsen and Shephard (2002). Although RV is an error free measure of volatility, it is susceptible to microstructure bias (Hansen & Lunde, 2006, Andersen *et al.*, 2011) as well as abrupt jumps in financial markets (Barndorff-Nielsen and Shephard 2004; Andersen *et al.*, 2012) To untangle the impact of the presence of rare jumps, Barndorff-Nielsen and Shephard (2004) introduced a bipower variation (BV) estimator which takes the form of cumulating product of adjacent absolute returns. Nonetheless, BV is sensitive to the presence of very small returns. Alternatively, based on the nearest neighbor truncation approach, Andersen *et al.*, (2012) proposed two jump-robust estimators, namely the minimum realized volatility (minRV) and the median realized volatility (medRV) to eliminate the noise in the volatility.

Apart from the volatility measures, an appropriate volatility model is crucial to ascertain the accuracy of the volatility forecast. Beforehand it is important to know the financial background that is embraced in the statistical models. For the past several decades, the informationally efficient market hypothesis (EMH) has been intensively studied theoretically and empirically (Fama, 1998; Malkiel, 2003) using financial markets data. An ideal efficient market suggests that all the relevant market information is reflected by the market price. As such, no investors are able to beat the markets via any asset selections or market timing strategies. There are two major approaches used to improve the analysis of EMH, namely the empirical methodology, and the theoretical framework that gives rise to the new definitions of EMH. The new definitions that complement the classical EMH are fractal market hypothesis (Peters, 1994), heterogeneous market hypothesis (Muller et al., 1993; Dacorogna, 1998) and adaptive market hypothesis (Lo, 2005). Heterogeneous market hypothesis (HMH) is among the new ideas that are recommended in the market efficiency literature. This concept has been introduced by Muller *et al.* (1993) and Dacorogna *et al.* (2001) in the foreign exchange and the stock markets. HMH suggests that the market participants are heterogeneous; thus, the same market information can be interpreted differently based on one's trading preference and opportunity. As such, under fluctuating price movements, volatility cascades ranging from low to high frequencies are created due to the diverse reactions from the heterogeneous market participants. The combination of these dissimilar volatilities (due to reaction times) is believed to produce a slow decaying autocorrelation function or long memory dependence property in financial markets. The long memory trait is commonly analyzed via the autoregressive fractionally integrated moving average models, ARFIMA (Andersen *et al.,* 2003; Barunik & Krehlik, 2016; Yap & Cheong, 2016). To give a comprehensive comparison, this study includes the discussion on the extension of ARFIMA to form the volatility model. Based on the HMH structure, the cascading volatility can be easily constructed using an additive hierarchical structure on the realized volatilities of various time horizons. The HMH heterogeneity has been studied with different approaches by researchers such as (Lux & Marchesi, 1999; Andersen & Bollerslev, 1997; Muller *et al.,* 1997; Cheong *et al.,* 2007; Cheong *et al.*, 2013; Corsi *et al.*, 2008; Corsi, 2009). Most of the aforementioned studies are conducted using high-frequency data (or intraday data) which are collected minutely from the daily trading activities in a specific financial market. With the information technology resources, the intraday data that reflect the heavy trading activities are readily available. Coupled with the concrete theoretical foundations on realized volatility (Andersen & Bollerslev, 1998; Blair *et al.,* 2001) the handiness of intraday data fuels the research interest on the use of high frequency data to improve the forecast performance in foreign exchange and the stock markets.

In this study, we attempt to use the extension of HAR and ARFIMA models to accommodate for the asymmetry volatility clustering as well as the asymmetric relationship between RV and the volatility of RV. The extended models are named as asymmetric skewed HARX (RV)-GJR-GARCH and ARFIMAX (RV)-GJR-GARCH which will be demonstrated using the S&P500 index. Besides using RV, we also include other alternatives such as tripower RV, minRV and medRV in both of the extended models. In addition, the RV's errors are considered as leptokurtic and asymmetrically distributed which follow a skewed student-*t* distribution. Comparing to the original models, the proposed model provides better in-sample as well as out-of-sample forecast evaluations. To complete this study, we illustrate a one-day-ahead value-at-risk determination using the estimated results. The remaining of this study is organized as follows: Section 2 provides the description of RV, tripower RV, minRV and medRV of volatility estimations and the volatility models; Section 3 discusses the empirical data and results and finally, Section 4 concludes the findings of the study.

Methodology

Integrated volatility estimation based on high- frequency data is used to measure the unobservable latent volatility. Let us consider a stochastic volatility process for the logarithmic prices, $p(t) = lnP(t)$ of an asset, $dp(t) = \mu(t)dt + \sigma(t)dW(t)$, where $\mu(t), \sigma(t)$ and $W(t)$ are the drift, volatility and standard Brownian motion respectively. The $\mu(t)$ and $\sigma(t)$ may be time-varying, but they are assumed to be independent of $dW(t)$. The continuously compounded intraday return of day t with sampling frequency of N per day is $r_{t,j} = 100\left(\ln P_{t,j} - \ln P_{t,j-1}\right), j = 1, \dots, N$, $t = 1, \dots, T$ For an increasing sequence of m random partitions $\tau_0 = 0 \le \tau_1 \le \cdots \le \tau_m = t$, the quadratic variation is equivalent to the integrated variance, that is, $\lim_{m\to\infty} \sum_{i=0}^{m-1}\left(p_{\tau_{i+1}} - p_{\tau_i}\right)^2 = \int_0^t \sigma^2(t)dt.$ Under this condition, the integrated variance can be estimated consistently by the RV, (Andersen & Bollerslev, 1998) $\sigma_{AB,t}^2 = \sum_{j=1}^{N} r_{t,j}^2$. However, with the presence of abrupt jumps, the RV is no longer a consistent estimate for the integrated variance.

Jump-Robust Volatility Estimators

In order to overcome the noisiness of the volatility, we adopt the tripower variation estimator (Barndorff-Nielsen and Shephard, 2002) as follows:

$$TV_t = MPV_{3,t}(i = 3, q = 2) = \mu_{2/3}^{-3} \frac{t}{t-2} \sum_{j=1}^{t-2} |r_j|^{2/3} |r_{j+1}|^{2/3} |r_{j+2}|^{2/3} \quad (1)$$

where *i* and *q* are positive integers with the relationship *i* > *q*/2 with a finite sample correction of $\left(\frac{t}{t-i+1}\right)$. In general, the tripower variations smoothen out the abrupt jumps by averaging the adjacent returns. The term *i* represents the window size of return blocks and q indicates the desired power variation of volatility. For i.i.d price changes, $\mu_{q/i} = 2^{\frac{q}{2i}}\Gamma[(q/i + 1)/2]/\Gamma[1/2]$, where $\Gamma[.]$ is a gamma function. It is worth noting that although TV is able to smooth the impact of a jump by multiplying two or more consecutive returns, it is not able to reduce the magnitude of two or more consecutive jumps. In addition, it is also sensitive to the very small returns, which subsequently leads to bias. Alternatively, Andersen *et al.*, (2012) proposed two estimators based on the minimum (minRV) and median (medRV) operators using the nearest neighbor truncation (NTT) approach, stated as follows:

$$minRV_t = \frac{\pi}{\pi - 2} \frac{t}{t-1} \sum_{j=1}^{t-1} \left[\min(|r_{t,j}|, |r_{t,j+1}|)\right]^2 \quad (2)$$

$$medRV_t = \frac{\pi}{6-4\sqrt{3}+\pi}\frac{t}{t-1}\sum_{j=2}^{t-1}\left[med(|r_{t,j-1}|,|r_{t,j}|,|r_{t,j+1}|)\right]^2 \quad (3)$$

The minimum realized volatility (minRV) eliminates a jump for a given block of two consecutive returns and it is computed based on the adjacent diffusive returns, whereas the median realized volatility (medRV) uses the median operator to square the median of three consecutive absolute returns. As a comparison, TV smoothen a possible jump whereas NTT estimators eliminate it from the block of returns. Andersen *et al.* (2012) showed that the NTT estimators are more efficient and robust under the presence of jumps.

The Volatility Models

The asymmetric skewed HARX (RV)-GJR-GARCH model

Specifically, the asymmetric skewed HARX (RV)-GJR-GARCH (1,1) model can be written as

$$\ln(RV_{i,t}^{d}) = \theta_{i,0}+\theta_{i,1}r_t + \theta_{i,d}\ln(RV_{i,t-1}^{day}) + \theta_{i,w}\ln(RV_{i,t-1}^{week}) + \theta_{i,m}\ln(RV_{i,t-1}^{month}) + a_{i,t} \quad a_{i,t} = \sigma_{i,t}\varepsilon_{i,t}, \quad \varepsilon_{i,t}\sim skewed-t \quad \sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}a_{i,t-1}^2+\alpha_{i,2}\left|a_{i,t}^2\right|I_t + \alpha_{i,3}\sigma_{i,t-1}^2 \quad (4)$$

where *RV* represents the type of RV with $RV_t^{week} = \frac{1}{5}\sum_{j=1}^{5} RV_{t-j}^{day}$ and $RV_t^{month} = \frac{1}{22}\sum_{j=1}^{22} RV_{t-j}^{day}$. Subscripts *i* = 1, 2, 3 and 4 indicates the standard RV, tripower variation (TV), minRV and medRV respectively. The $\sigma_{i,t}^2$ is interpreted as the volatility of RV and $I_t(\cdot)$ is an dummy variable for $a_{i,t} < 0$. For instance, when $\alpha_2 > 0$, negative (or positive) news contribute to greater (or smaller) magnitude of RV. The original asymmetric GJR threshold (Glosten et. al., 1993) specification is meant to capture the leverage effect in financial market. However in this study, this specification is used to explore the relationship between various RVs and their volatilities. For the next financial stylized fact, the *X* in the model indicates whether the risk-premium (risk-return tradeoff) exists in the time series under study. The returns are expected to have a positive correlation to the intensity of market volatility or risk. In other words, higher risk asset should offer higher returns in order to attract investors to hold it.

The asymmetric skewed ARFIMAX (RV)-GJR-GARCH model.

For asymmetric skewed *ARFIMAX (RV)-GJR-GARCH model*, the specifications are as follow:

$$\varphi(B)(1-B)^d\left(lnRV_{i,t}-\mu_i\right) = \psi(B)a_{i,t} \quad a_{i,t} = \sigma_{i,t}\varepsilon_{i,t}, \quad \varepsilon_{i,t}\sim skewed-t \quad \sigma_{i,t}^2 = \beta_{i,0} + \beta_{i,1}a_{i,t-1}^2+\beta_{i,2}\left|a_{i,t}^2\right|I_t + \beta_{i,3}\sigma_{i,t-1}^2 \quad (5)$$

where $(1-B)^d$ denotes the fractional differencing operator, $\varphi(B)$ and $\psi(B)$ are backshift polynomials with respect to the autoregressive and moving average operators. The asymmetric GJR-GARCH specifications are able to capture all the financial stylized facts as the aforementioned HARX model.

For both the models, the volatility innovations are assumed to be leptokurtic and asymmetrically distributed under a skewed student-*t* distribution (Lambert & Laurent, 2001), $\varepsilon_{i,t}|\Omega_{t-1}\sim \text{skew}-t(0,1;v,k)$. The skewed student-t density function can be written as

$$f(\varepsilon_{i,t};v,k) = \begin{cases} \frac{\Gamma[\frac{v+1}{2}]}{\Gamma[\frac{v}{2}]\sqrt{\pi(v-2)}}\left(\frac{2s}{k+k^{-1}}\right)\left(1+\frac{s\varepsilon_{i,t}+m}{v-2}k\right)^{-(\frac{v+1}{2})} & \text{if } \varepsilon_{i,t}<-ms^{-1} \\ \frac{\Gamma[\frac{v+1}{2}]}{\Gamma[\frac{v}{2}]\sqrt{\pi(v-2)}}\left(\frac{2s}{k+k^{-1}}\right)\left(1+\frac{s\varepsilon_{i,t}+m}{v-2}k\right)^{-(\frac{v+1}{2})} & \text{if } \varepsilon_{i,t}\geq-ms^{-1} \end{cases} \quad (6)$$

with v and k are the tail and asymmetry parameters respectively where $s=\sqrt{k^2+k^{-2}-m^2-1}$ and $m=\frac{k-k^{-1}}{\Gamma\{[(\frac{v-1}{2})]\sqrt{v-2}\Gamma[\frac{v}{2}]\sqrt{\pi}\}}$. Overall, the estimated parameter vector for HARX is $\widehat{\Theta}(\boldsymbol{\theta},\boldsymbol{\alpha},v,k)$ where $\boldsymbol{\theta}=(\theta_0,\ \theta_1,\ \theta_d,\ \theta_w,\ \theta_m)$ and $\boldsymbol{\alpha}=(\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3)$. Assuming ARFIMA $(1,d,0)$, the estimated parameter vector is $\widehat{\Phi}(\boldsymbol{\varphi},\ \boldsymbol{\beta},\ v,\ k)$ with $\boldsymbol{\varphi}=(\varphi_0,\ \varphi_1,\ d)$ and $\boldsymbol{\beta}=(\beta_0,\ \beta_1,\ \beta_2,\ \beta_3)$. Using the Ox-G@RCH, the estimations are conducted using the simulated annealing maximum likelihood (MaxSA) as there may be possibly more than one local extrema which may not be smoothen.

For model diagnostic, the Ljung-Box serial correlations are used to examine the standardized and squared standardized residuals under the null hypothesis of uncorrelated series. Next, the model selections are based on three information criteria namely the Akaike, Schwarz and Hannan-Quinn that are evaluated from the adjusted average log likelihood function.

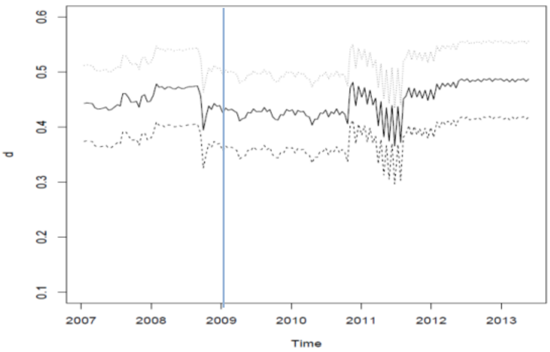
After the in-sample forecast evaluation, the out-of-sample forecast evaluations are conducted based on a rolling fixed sample size of T=1623 for h one-day ahead forecast where $h=1,2,\dots,H$ and H is fixed at 120. The various one-day-ahead logarithmic RV forecasts for HARX and ARFIMAX models are computed based on the parameter vector $\widehat{\Theta}^{(t)}(\boldsymbol{\theta}^{(t)},\boldsymbol{\alpha}^{(t)},v^{(t)},k^{(t)})$ and $\widehat{\Phi}^{(t)}(\boldsymbol{\varphi}^{(t)},\boldsymbol{\beta}^{(t)},v^{(t)},k^{(t)})$, which are re-estimated every day for *t* = T+1, …, T+120.

In order to evaluate the best out-of-sample forecast, we have selected root mean squared error (RMSE = $\frac{1}{H}\sum_{h=t+1}^{t+H}\left(\sigma_{Actual,h}^2-\sigma_{Forecast,h}^2\right)^2$), mean absolute error (MAE= $\frac{1}{H}\sum_{h=t+1}^{t+H}\left|\sigma_{Actual,h}^2-\sigma_{Forecast,h}^2\right|$) and Mincer, Zarnowitz (MZ) regression, $\sigma_{Actual,h}^2=\varphi_0+\varphi_1\sigma_{Forecast,h}^2+u_h$ to indicate the power of predictability. The MAE is better than RMSE due to its mild responses to large errors whereas the MZ is also robust (Meddahi, 2001) to noise in the forecasted volatility. This study follows the robustness definition by Patton (2011) where the models ranking should be consistent no matter what types of proxies are being used as true values in the forecast evaluations. Although there are other more advanced forecast evaluation methods (Diebold & Mariano, 1995; White, 2000; Hansen, 2005), we focus on the aforementioned measurements which evaluate the deviation between forecasts and realizations. In order to provide a fair and objective forecast evaluation, the performance of the models are examined using RV, TV, minRV and medRV as the proxy of the actual volatility. A simple scoring scheme is used to facilitate the ranking of the forecast performance amongst these models.

Empirical Study

In this specific study, we are interested to explore the volatility behavior of the U.S. stock market during the subprime mortgage crisis. For empirical study, we use the U.S. S&P500 index from the Bloomberg database spanning from 1st Feb 2008 to 27th February 2015 with a total of 1768 observations. This includes the out-of-sample forecast evaluations data from 31st July 2014 to 27th February 2015. It is noted that we have included the subprime mortgage crisis period started from early 2008 to ensure that the empirical data is highly volatile for possible jumps in the series.

In order to uncover this crisis, we conduct a dynamic long memory evaluation of S&P500 RV starting from the year 2007. The long memory parameter is estimated using the rolling fixed window wavelet maximum likelihood estimation approach by Jensen (2000), with the indicator of the fractional integrated differencing parameter, d. The fixed time window is set to 1024 observations with four year data (2003-2007) as an initiation. The d is interpreted as stationary long memory for $d \in (0, 0.5)$ and nonstationary for $d \in (0.5, 1.0)$.



Note: The fixed time window is set at 1024 observations

Figure 1. Rolling estimated *d*

Figure 1 shows that the values of *d* are within the range 0.4 to 0.5. It is found that, there is a drastic drop of *d* in September of 2008 when the Lehman Brothers filing for U.S. Federal government bailout. This is the moment when the short horizon investors are dominating the market whereas the long horizon investors are either withdrawn from the markets or participating in the short horizon trading activities. In other words, the liquidity of the trading interactions amongst different horizon investors is lost, thus the heterogeneity of investors is also removed under severe selling pressure.

Table 1

Descriptive statistics for various logarithmic RVs

statistics	LOG(RV)	LOG(TV)	LOG(minRV)	LOG(medRV)
Mean	-0.258622	-0.703513	-0.752105	-0.731071
Median	-0.328343	-0.817589	-0.879647	-0.852047
Std. Dev.	1.212779	1.184659	1.189649	1.186975
Skewness	0.544305	0.670947	0.672256	0.671845
Kurtosis	3.363573	3.670900	3.631880	3.679232
Jarque-Bera	89.02456*	152.1157*	149.1553*	153.2020*

Note: Jarque-Bera statistic = $\frac{T}{6}\left(skewness + \frac{(Kurtosis-3)^2}{4}\right)$
* indicates 5 % level of significance.

Table 1 gives a quick glance at the descriptive statistics of all the logarithm RVs. It is found that the expected value of RVs are non-zero, slightly positively skewed to right with kurtosis larger than three as compared to a standardized normal distribution. A normality test using Jarque-Bera test found that all the series are statistically deviated from the normal distribution at 5 % level of significance. As a summary, the logarithmic RV series are fat-tailed, slightly skewed to the right compared to a normal distribution. These statistical behaviors should be included in the model specifications.

Estimation Results

Table 2 and 3 report a total of 8 ARFIMAX and HARX models based on Eqs. (4) and (5). The errors are assumed to follow a skewed student-*t* distribution, as the preliminary analysis indicates the presence of heavy-tailed and positive skewness in the volatility.

Table 2

Estimation for ARFIMAX (1,*d*,0)-GJR-GARCH(1,1) Skewed-*t*

ARFIMA:	RV	TV	minRV	medRV
ϕ0	0.645870** (0.24181)	0.309088 (0.25322)	0.189198 (0.28684)	0.181611 (0.21582)
Risk premium, ϕ1	-4.894126** (0.86311)	-3.195552** (0.59343)	-3.510232 ** (0.66058)	-3.231450** (0.59656)
Long memory, d	0.555783** (0.023108)	0.602821** (0.023819)	0.580117** (0.022687)	0.591062** (0.024674)
Lag return, ϕ2	-0.271834** (0.031961)	-0.107700** (0.032981)	-0.117337 ** (0.031505)	-0.103338** (0.033149)
GARCH:				
β0	0.116370 ** (0.048529)	0.022834** (0.0091010)	0.024547 ** (0.010941)	0.018891** (0.0095447)
ARCH effect, β1	0.157004** (0.042488)	0.095153** (0.022609)	0.094465 ** (0.023431)	0.085920** (0.025173)
GJR effect, β2	-0.083022** (0.044010)	-0.087156 ** (0.036880)	-0.099270 ** (0.037163)	-0.076540 ** (0.036693)
GARCH effect, β3	0.639050** (0.12722)	0.869764 ** (0.033606)	0.880009** 0.035856)	0.889406 ** (0.038465)
Tail effect				
Positive skewed, k	0.220577* (0.039876)	0.146946 ** (0.037063)	0.101912 ** (0.039241)	0.132979 ** (0.039928)
Heavy tail, v	10.790454** (3.3572)	9.961764** (2.2595)	12.057063** (3.4116)	11.187558** (2.8920)
Selection:				
AIC	1.992949	1.511084	1.634400	1.548037
SIC	2.026189	1.544323	1.667639	1.581276
HIC	1.992874	1.511009	1.634324	1.547961
Diagnose:				
Q(10) on Std Res	6.35580 [0.7038473]	13.7781 [0.1304392]	18.8531 [0.0264715]*	15.8333 [0.0704461]
Q(10) on Squared Std Res	3.22014 [0.9197926]	10.2456 [0.2482117]	13.0025 [0.1117648]	8.51526 [0.3848210]

* and ** indicate 5 % and 1 % level of significance respectively
(·) represents the standard error of the estimated parameter
[·] represents the p-value

The estimation of asymmetric skewed HARX-GARCH models show that the heterogeneous autoregressive components (θ_d θ_w and θ_m) for the past daily, weekly and monthly volatilities are all different from zero at 5 % level of significance. This supports the heterogeneous market hypothesis where the markets consist of heterogeneous market participants with different time horizon investment preferences.

Table 3

Estimation for HARX-GJR-GARCH(1,1) Skewed-*t*

HARX	RV	TV	minRV	medRV
θ_0	-0.166106** (0.026439)	-0.074069** (0.015601)	-0.085501** (0.017246)	-0.074733** (0.018125)
Risk premium, θ_1	-5.731659 ** (0.97988)	-4.753559 ** (0.67091)	-5.211187** (0.69947)	-4.767024** (0.65210)
Lagged one Daily effect, θ_{d-1}	0.161962** (0.036788)	0.430673** (0.029338)	0.385727** (0.028690)	0.421489 ** (0.028682)
Lagged two Daily effect, θ_{d-2}	0.157122** (0.033903)			
Lagged one Weekly effect, θ_{w-1}	0.443923 ** (0.084823)	0.346255 ** (0.039757)	0.381244 ** (0.040508)	0.347867 ** (0.040319)
Lagged one Monthly effect, θ_{m-1}	0.489938** (0.086112)	0.203454** (0.029031)	0.210315** (0.030910)	0.214368** (0.030181)
GARCH				
α_0	0.074653 (0.069306)	0.011817 ** (0.0074767)	0.011142 (0.0071771)	0.007775 (0.0074251)
ARCH effect, α_1	0.156181** (0.063363)	0.082213** (0.022998)	0.080794** (0.022790)	0.073186 ** (0.027142)
GJR effect , α_2	-0.041194 (0.042117)	-0.062108** (0.029357)	-0.070183** (0.025805)	-0.061248** (0.027722)
GARCH effect, α_3	0.715157** (0.19896)	0.907050** (0.037445	0.919537 ** (0.033410)	0.931037** (0.044926)
Tail effect				
Positive skewed, *k*	0.219119 ** (0.040801)	0.142711** (0.042693)	0.089598** (0.040763)	0.131946** (0.043622)
Heavy tail, *v*	10.600026 ** (3.1635)	9.461295** (2.0816)	11.260353** (3.0831)	10.721542** (2.7170)
Selection				
AIC	2.015484	1.504473	1.624983	1.540621
SIC	2.055371	1.541036	1.661547	1.577185
HIC	2.015376	1.504382	1.624892	1.540530
Diagnose				
Q(10) on Std Res	8.73434 [0.3652005]	16.2113 [0.0937408]	10.8918 [0.3660074]	14.3535 [0.1574690]
Q(10) on Squared Std Res	3.84489 [0.8708394]	8.75900 [0.3630326]	9.30398 [0.3173055]	9.79238 [0.2799015]

* and ** indicate 5% and 1% level of significance respectively
(·) represents the standard error of the estimated parameter
[·] represents the p-value

Interestingly, it is noted that HARX models with realized volatility represented by TV, minRV and medRV show that the past daily volatility contributes the strongest impact to the logarithmic realized daily volatility, followed by the weekly and monthly volatility. This is in line with the widely acceptable notion that the most recent past volatility should have the largest impact on the current volatility. However, HARX model with the standard RV volatility representation (RV) shows the reverse impact, which may be caused by the un-smoothen noisy RV. For the risk premium coefficient, θ_1, all of the HARX models indicate positive correlation between the volatility and the negatively expected return (logarithmic volatility is in negative value). This is rather reasonable as the asset with a higher risk should offer a higher return in order to encourage the investors to own it. In the asymmetric GJR-GARCH components, the coefficient α_2 for the models with realized volatility represented by TV, minRV and medRV are significant at 5 %. This shows that there is a significant impact due to the asymmetric volatility, and therefore, it is necessary to consider the asymmetric GJR-GARCH component. For skewness of the various RV innovations, the coefficient *k's* are all positively skewed and all the tail parameters, *v's* exhibit fatter tails than normal distribution with the values around 10 degrees of freedom. In other words, the innovations are heavy-tailed and positively skewed.

For the asymmetric skewed ARFIMAX-GJR-GARCH model, the fractional difference parameter, *d's* are all significantly different from zero, which indicate the presence of long memory volatility. Apart from this, the risk premium coefficient ϕ_2, the coefficients related to the volatility of RV $\beta's$, and the skewed-*t* distribution indicate similar results as the HARX models.

For the model diagnostic, all the models under skewed-*t* innovations failed to reject the Ljung-Box serial correlations for standardized squared innovations. However for standardized innovation of ARFIMAX (minRV), the test is rejected at 1 % level of significance. This finding indicates that the minRV representation does not statistically fit well in the ARFIMAX model. The model selections are based on AIC, BIC and HIC. According to these information criteria, the models under the assumption of skewed-*t* distributed innovation outperform the normally distributed innovation models. Also, all the asymmetric skewed HARX models perform better than the asymmetric skewed ARFIMAX models. As a summary, the asymmetric skewed HARX is the most preferable model compared to the others in the estimation. However, there is no guarantee that this result will persist in the out-of-sample forecast evaluations due to other factors such as over-parameterization issue and unforeseen structural changes in the series (Hong *et al.*, 2004).

Forecast evaluations

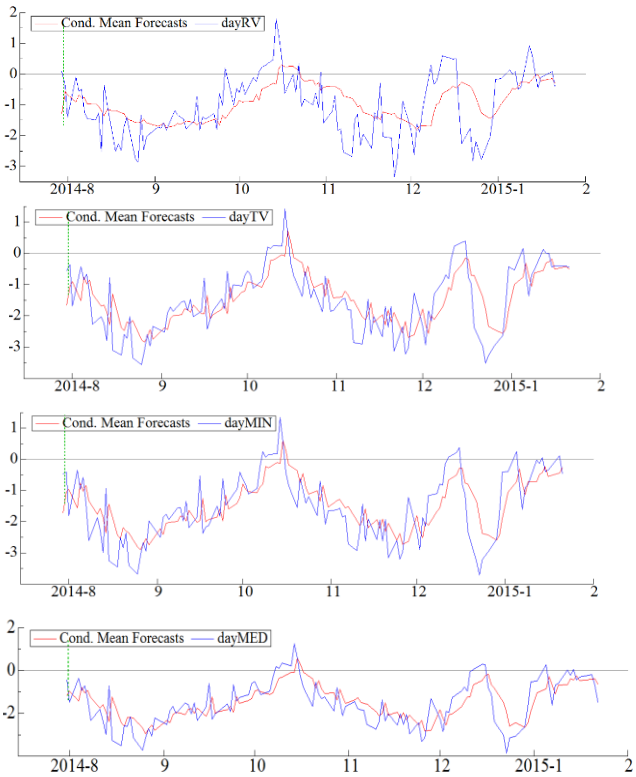


Figure 2. Out-of-sample forecasts for the asymmetric skewed HARX-GJR-GARCH

In order to provide reliable out-of-sample forecast evaluations, the actual volatility is alternately represented by the proxies of RV, TV, minRV and medRV. For each of these proxies, a simple scoring rule is applied to rate the forecast performance amongst the competing models. Under this rule, the best model will be given 8 points. The mark reduces point by point, and eventually the worst model is given 1 point. The scores under the eight models with different volatility proxies will be added to a final score for the ranking purposes. Table 4 and Figure 2 report the forecast evaluations for RMSE, MAE and MZ test for all the models.

Table 4

Forecast evaluations using MAE, RMSE and MZ regression test

MAE	**Actual volatility proxy**				
Model	**RV**	**TV**	**minRV**	**medRV**	**rank**
ARFIMAX – RV	0.83492	0.93059	0.95904	0.94392	8
-TV	0.88964	0.86509	0.87844	0.86136	6
- minRV)	0.89288	0.86449	0.87696	0.85995	5
- medRV	0.89531	0.86595	0.87801	0.86096	7
HARX-RV	0.70273	0.84172	0.88048	0.85646	4
-TV	0.61875	0.57111	0.59585	0.57244	**1**
- minRV)	0.64290	0.57399	0.59718	0.57499	**3**
- medRV	0.63243	0.56652	0.59604	0.56900	**1**
RMSE	Actual volatility proxy				
Model	RV	TV	minRV	medRV	rank
ARFIMAX – RV	1.00015	1.13628	1.16986	1.14898	8
-TV	1.10320	1.04030	1.05685	1.03898	5
- minRV)	1.10721	1.03915	1.05517	1.03745	4
- medRV	1.11069	1.04051	1.05627	1.03863	6
HARX-RV	0.87754	1.06303	1.10554	1.08465	7
-TV	0.77161	0.70561	0.74263	0.71675	**3**
- minRV)	0.79922	0.70379	0.73798	0.71220	**1**
- medRV	0.78940	0.70265	0.73918	0.71302	**1**
MZ test (R^2)	Actual volatility proxy				
Model	RV	TV	minRV	medRV	rank
ARFIMAX - RV	0.0873	0.1147	0.1134	0.1161	5
-TV	0.0638	0.0884	0.0866	0.0906	8
- minRV)	0.0678	0.0935	0.0917	0.0956	6
- medRV	0.0664	0.0916	0.0898	0.0937	7
HARX-RV	0.2669	0.2459	0.2317	0.2358	4
-TV	0.5759	0.5560	0.5277	0.5444	**3**
- minRV)	0.5651	0.5566	0.5290	0.5462	**1**
- medRV	0.5756	0.5582	0.5279	0.5455	**1**

MZ test: adjusted R^2 under simple linear regression.

Overall, the jump-robust volatilities (TV, minRV and medRV) show better scores than RV in all the evaluations. This is an expected outcome because RV is nosier than the other volatility proxies. The HARX models outperform the ARFIMAX models under the similar volatility proxies. In other words, the HARX specifications under the heterogeneous market hypothesis are better at explaining the fluctuation of market prices. Overall, the first three best ranked models are consistent under the evaluations of MSE, MAE and MZ tests. The HARX model based on medRV is ranked as the best, followed by the minRV and TV. This is parallel to the definition of robustness by Patton (2011) whereby the forecast performance ranking is consistent regardless of the proxy used in the evaluations. It is worth to note that the determination of MZ test improves to approximately 0.5600 under the representation of TV, minRV and medRV for the HARX models. In other words, the forecasted volatility is able to explain approximately 56 % of the variation in the actual volatility. The ARFIMAX models on the other hand, only explain around 10 % of the variation in the actual volatility.

Market Risk Determination Using Value-At-Risk

For market risk determination, we compute the value-at-risk using both the HARX and ARFIMAX models based on RV, TV, minRV and medRV models. The value-at-risk (VaR) is one of the famous market risk indicators (Jorion, 2006) in the actuarial industries. Following the probabilistic framework by Tsay (2010), let $\Delta r(\tau)$ be the change in value of the returns in stocks market from time t to $t+\tau$ for a market. Denote the cumulative distribution function (CDF) of $\Delta r(\tau)$ by $F_\tau(x)$, the individual VaR of a long position over the time horizon τ with probability α is defined as

$$F_{long}(\text{VaR}_i)= P[\Delta r_i(\tau) \leq \text{VaR}_i]=\alpha . \quad (7)$$

For example, under the asymmetric skewed HARX-GARCH estimation, the long financial position of single market q% quantile is written as

$$quantile(k) = r(k) + \left(D_{q\%} \times \hat{\sigma}(k)\right) \quad (8)$$

where $\hat{\sigma}(k)$ represents the conditional volatility forecast for k-day ahead. The r can be forecasted using the ARIMA model under the assumption of D distribution. The D can be assumed to be a normal or a student-t distribution in this specific illustration. Finally, the VaR can be quantified as a product of the capital of investment and the quantile at a specific level of confidence within a predefined time horizon.

Under a long position trading, an investor buys a stock, holds it while it appreciates, and eventually sells it for profit. He encounters risk when the price of the stock plunges, which occurs at the left tail of the return distribution. Suppose that an investor holds a long financial position of the S&P500 stock market with a capital of $1 million. The 5% quantile for one-day ahead asymmetric skewed HARX (TV)-GJR-GARCH for the returns with normal and student-t distributions are Normal return:

$$quantile(1)_{normal} = r(1) + z_{0.05} \times \hat{\sigma}_{TV}(1)$$

$$0.002568 + (-1.64485) \times (0.595386) = -0.97675$$

Student-t return: $quantile(1)_{student-t}$

$$= r(1) + t_{v=4.861553} \times \hat{\sigma}_{TV}(1).$$

$$= 0.003073 + (-2.13185) \times (0.595386) = -1.26620$$

It is understood that the negative sign in $quantile(1)$ indicates a loss, which is located at the left tail distribution. The normal VaR with probability 0.05 is 0.97675×$1000000 = $9767.5 whereas the student-t VaR is $12662.0. These results show that with probability 95%, the potential loss of holding this position for the next day (1 day horizon) is $9767.5 and $12662 respectively for these return distributions. The student-t assumption indicates a higher VaR compared to the normal assumption. In other words, making an inappropriate parametric distribution assumption against the empirical student-t distribution often faces the underestimation issue in VaR determination. Similarly, the VaR with probability 0.01 can be computed using the same procedures. Table 5 shows the overall results of VaR evaluations for all the volatility models.

Table 5

Value-at-risk for various types of RV

	Type of RV			
Value-at-risk	RV	TV	minRV	medRV
VaR 5% - Normal	$12366.8	$9767.5	$9787.4	$9588.4
VaR 5% - Student-*t*	$16030.8	$12662.0	$12687.8	$12429.8
VaR 1% - Normal	$17501.2	$13825.1	$13853.2	$13571.7
VaR 1% - Student-*t*	$28199.1	$22278.1	$22323.4	$21870.0

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Conclusion

This study introduces the asymmetric skewed ARFIMAX and HARX-GJR-GARCH models in the S&P 500 index. Besides the standard realized volatility, we examine these models with the jump-robust volatilities such as tripower realized volatility and the nearest neighbor truncation realized volatility. The extended HARX-GJR-GARCH models are capable to capture the risk premium, asymmetric volatility of the realized volatility, skewed and heavy-tailed innovations. They perform better in the estimation and the out-of-sample forecast evaluations compared to their counterpart, the ARFIMAX-GJR-GARCH models. The empirical study shows that the jump-robust realized volatilities outperform the standard realized volatility in the forecast evaluations. As a conclusion, this study provides alternative models that are able to deal with high volatile market condition. In addition, the extended HARX-GJR-GARCH models are also in line with the financial framework of the heterogeneous market hypothesis. An illustration of value-at-risk shows that the forecasted results can be easily used in the market risk determination which provides very useful information for portfolio investments and risk management

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